Continuous Probability Distributions

Chapter 7

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Learning Objectives

LO1 List the characteristics of the uniform distribution.
LO2 Compute probabilities by using the uniform distribution.
LO3 List the characteristics of the normal probability distribution.
LO4 Convert a normal distribution to the standard normal distribution.
LO5 Find the probability that an observation on a normally distributed random variable is between two values.
LO6 Find probabilities using the Empirical Rule.
LO7 Approximate the binomial distribution using the normal distribution.
LO8 Describe the characteristics and compute probabilities using the exponential distribution.

LO1 List the characteristics of the uniform distribution.

The Uniform Distribution

The uniform probability distribution is perhaps the **simplest distribution for a continuous random variable**.

This distribution is **rectangular in shape** and is defined by minimum and maximum values.

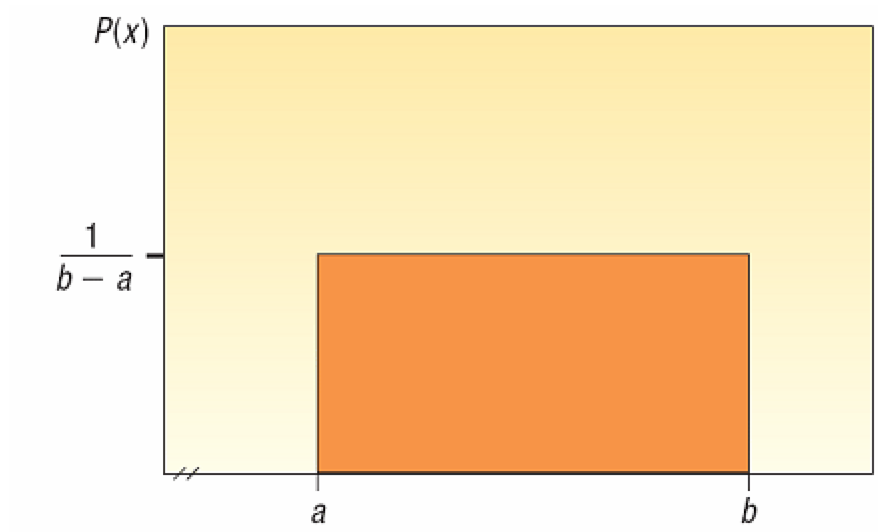


CHART 7–1 A Continuous Uniform Distribution

The Uniform Distribution

The Continuous Uniform Distribution:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where

$f(x)$ = value of the density function at any x value

a = lower limit of the interval

b = upper limit of the interval

The Uniform Distribution – Mean and Standard Deviation

MEAN OF THE UNIFORM DISTRIBUTION

$$\mu = \frac{a + b}{2} \qquad [7\text{–}1]$$

STANDARD DEVIATION OF THE UNIFORM DISTRIBUTION

$$\sigma = \sqrt{\frac{(b - a)^2}{12}} \qquad [7\text{–}2]$$

UNIFORM DISTRIBUTION

$$P(x) = \frac{1}{b - a} \quad \text{if } a \leq x \leq b \text{ and 0 elsewhere} \qquad [7\text{–}3]$$

LO2 Compute probabilities by using the uniform distribution.

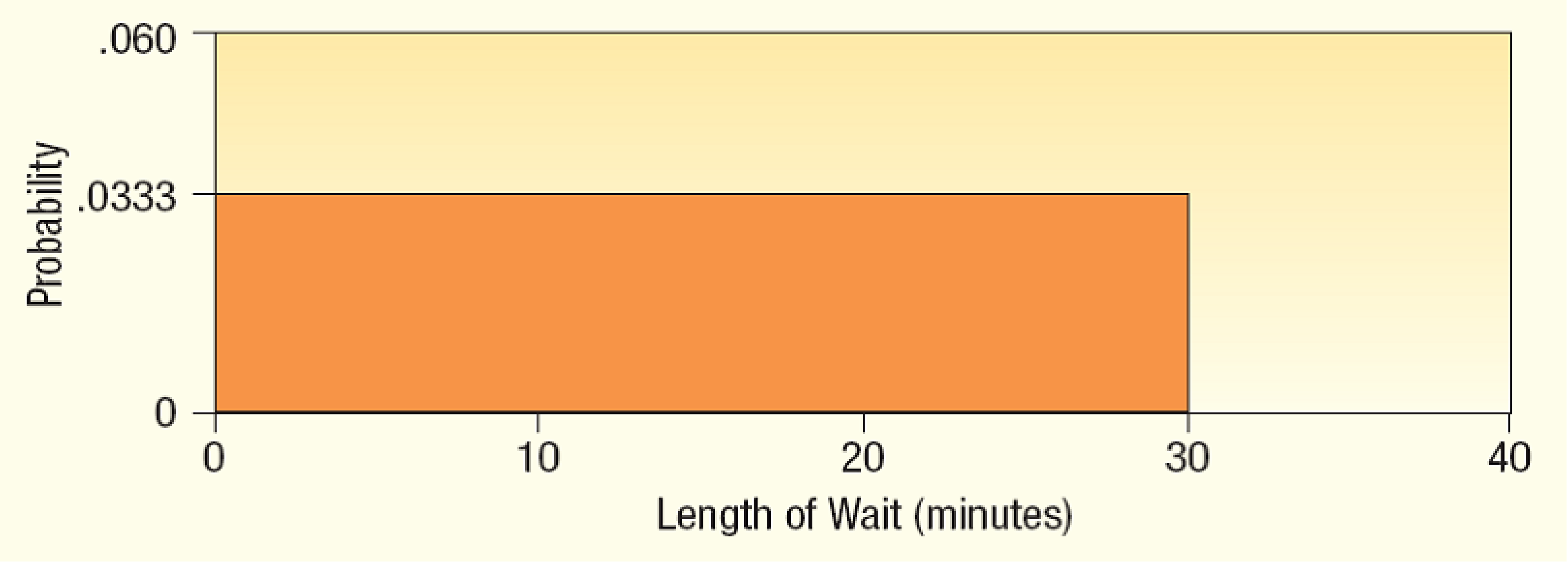
The Uniform Distribution - Example

Southwest Arizona State University provides bus service to students while they are on campus. A bus arrives at the North Main Street and College Drive stop every 30 minutes between 6 A.M. and 11 P.M. during weekdays. Students arrive at the bus stop at random times. The time that a student waits is uniformly distributed from 0 to 30 minutes.

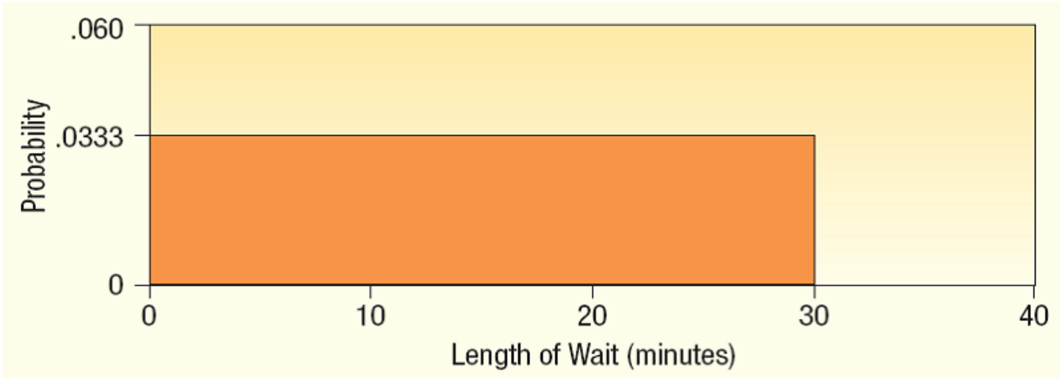
1. Draw a graph of this distribution.
2. Show that the area of this uniform distribution is 1.00.
3. How long will a student "typically" have to wait for a bus? In other words what is the mean waiting time? What is the standard deviation of the waiting times?
4. What is the probability a student will wait more than 25 minutes
5. What is the probability a student will wait between 10 and 20 minutes?

The Uniform Distribution - Example

1. Graph of this distribution.



The Uniform Distribution - Example



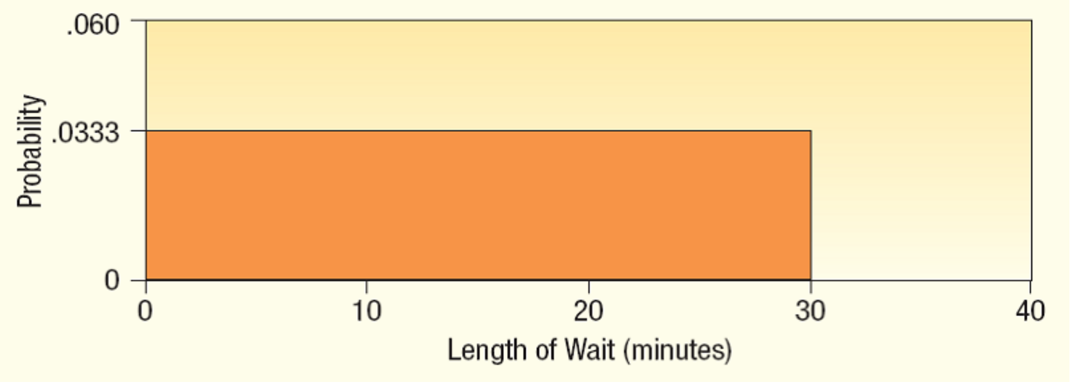
2. Show that the area of this distribution is 1.00

The times students must wait for the bus is uniform over the interval from 0 minutes to 30 minutes, so in this case *a* is 0 and *b* is 30.

$$\text{Area} = (\text{height})(\text{base}) = \frac{1}{(30-0)}(30-0) = 1.00$$

LO2

The Uniform Distribution - Example



3. How long will a student "typically" have to wait for a bus? In other words what is the **mean waiting time**?

$$\mu = \frac{a+b}{2} = \frac{0+30}{2} = 15$$

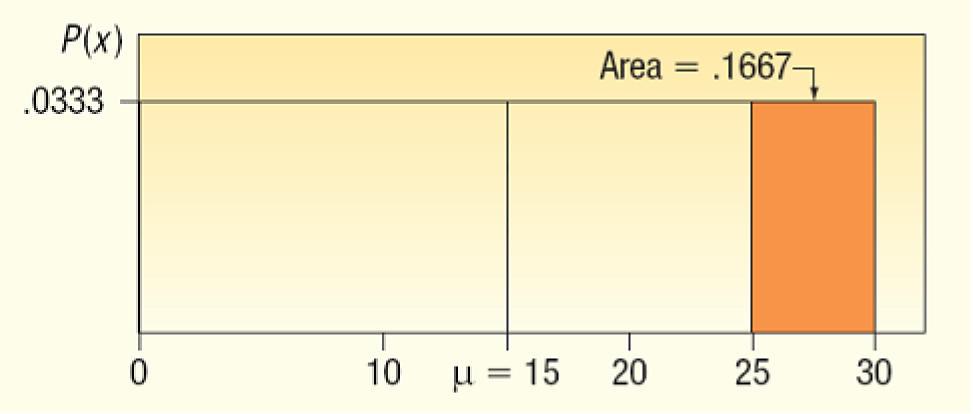
What is the **standard deviation** of the waiting times?

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(30-0)^2}{12}} = 8.66$$

The Uniform Distribution - Example

4. What is the probability a student will wait **more than 25 minutes**?

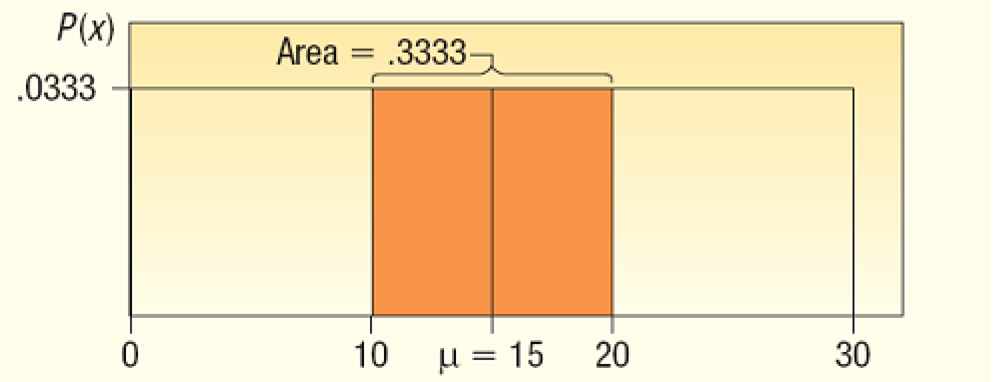
$$P(25 < \text{Wait Time} < 30) = (\text{height})(\text{base})$$
$$= \frac{1}{(30-0)}(5)$$
$$= 0.1667$$



The Uniform Distribution - Example

5. What is the probability a student will wait **between 10 and 20** minutes?

$$\begin{aligned}P(10 < \text{Wait Time} < 20) &= (\text{height})(\text{base})\\ &= \frac{1}{(30-0)}(10)\\ &= 0.3333\end{aligned}$$

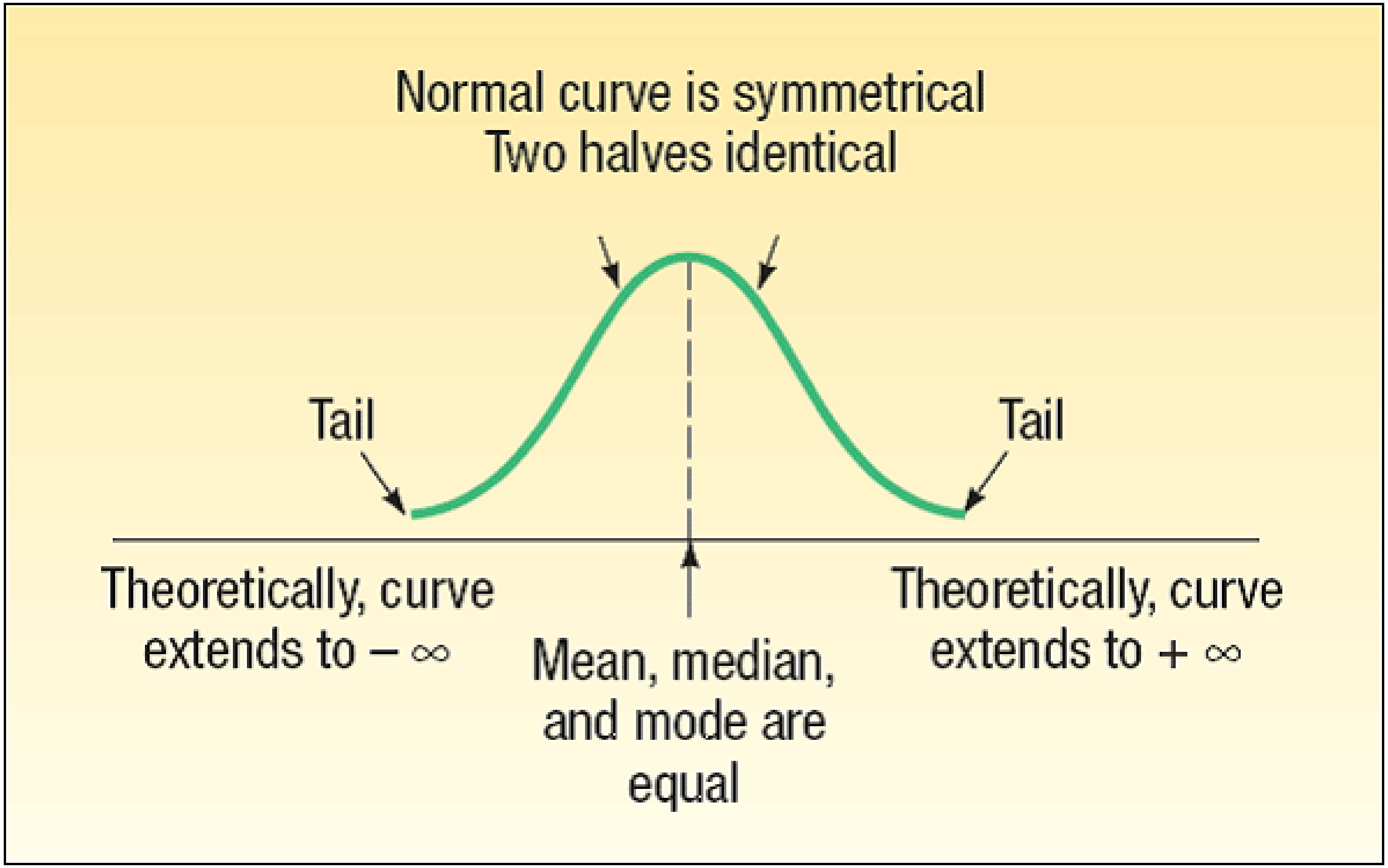


LO3 List the characteristics of the normal probability distribution.

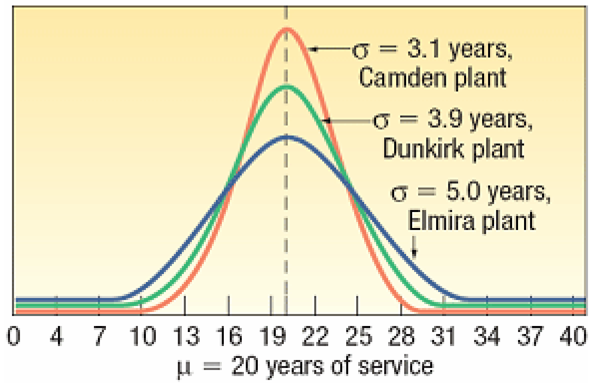
Characteristics of a Normal Probability Distribution

1. It is **bell-shaped** and has a single peak at the center of the distribution.
2. It is **symmetrical** about the mean
3. It is **asymptotic:** The curve gets closer and closer to the *X*-axis but never actually touches it. To put it another way, the tails of the curve extend indefinitely in both directions.
4. The location of a normal distribution is determined by the mean, μ, the dispersion or spread of the distribution is determined by the standard deviation, σ .
5. The arithmetic **mean, median**, and **mode are equal**
6. The total **area under the curve is 1.00**; half the area under the normal curve is to the right of this center point, the mean, and the other half to the left of it.

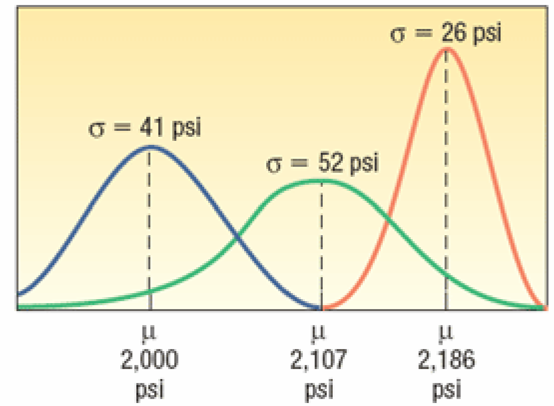
The Normal Distribution - Graphically



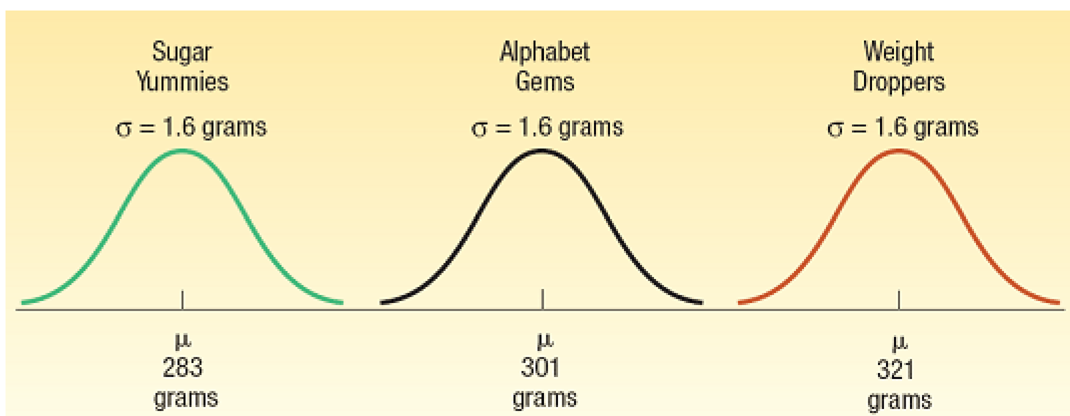
The Family of Normal Distribution



Equal Means and Different Standard Deviations



Different Means and Standard Deviations



Different Means and Equal Standard Deviations

LO4 **LO4** Convert a normal distribution to the standard normal distribution.

The Standard Normal Probability Distribution

- The standard normal distribution is a normal distribution with a **mean of 0** and a **standard deviation of 1** (great to have to apply the probability tables)
- It is also called the ***z* distribution**.
- Any normal distribution can be converted into standard normal distribution by getting the z-values
- A ***z*-value** is the signed distance between a selected value, designated *X*, and the population mean μ, divided by the population standard deviation, σ.
- The formula is:

$$z = \frac{X - \mu}{\sigma}$$

Areas Under the Normal Curve

E.g. Sugar Yummies
P(283<weight<285.4)?

1. Get z values:

- $(283-283)/1.6 = 0$
- $(285.4-283)/1.6 = 1.5$

z	0.00	0.01	0.02	0.03	0.04	0.05	...
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	
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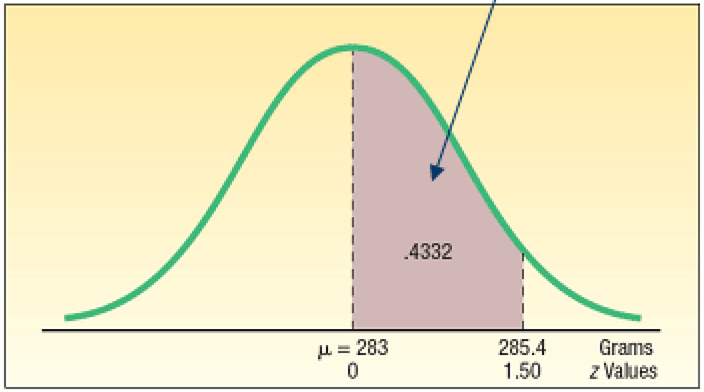


The Normal Distribution – Example

The weekly incomes of shift foremen in the glass industry follow the normal probability distribution with a **mean of $1,000** and a **standard deviation of $100**.

What is the ***z* value** for the income, let's call it *X*, of a foreman who earns **$1,100** per week? For a foreman who earns **$900** per week?

For $X = \$1,100$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1,100 - \$1,000}{\$100} = 1.00$$

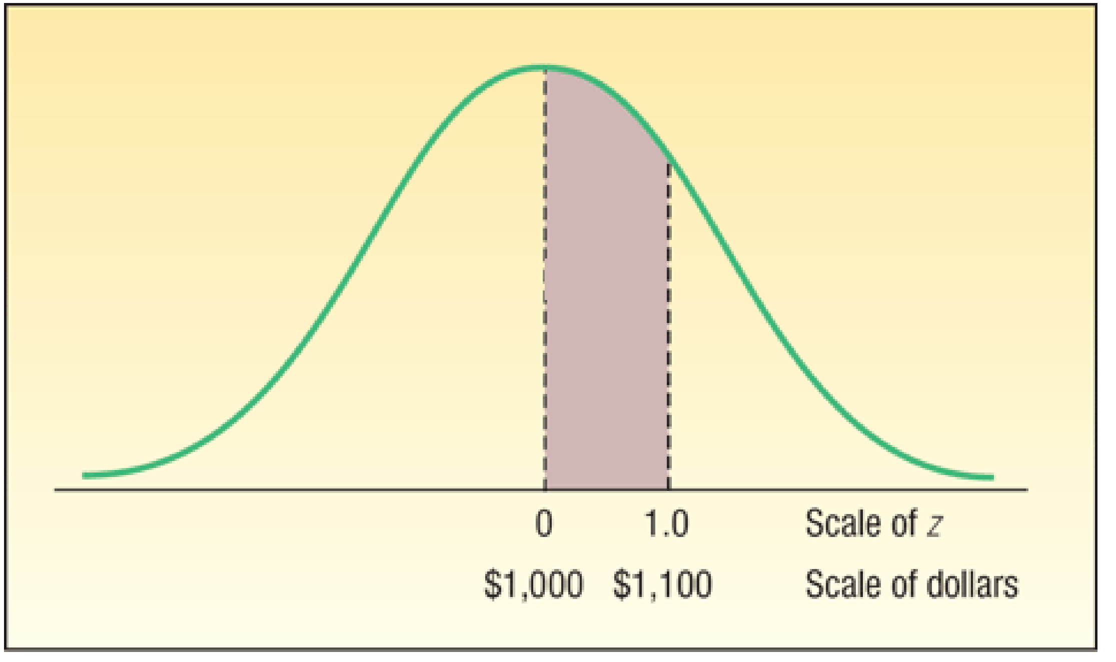
For $X = \$900$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$900 - \$1,000}{\$100} = -1.00$$

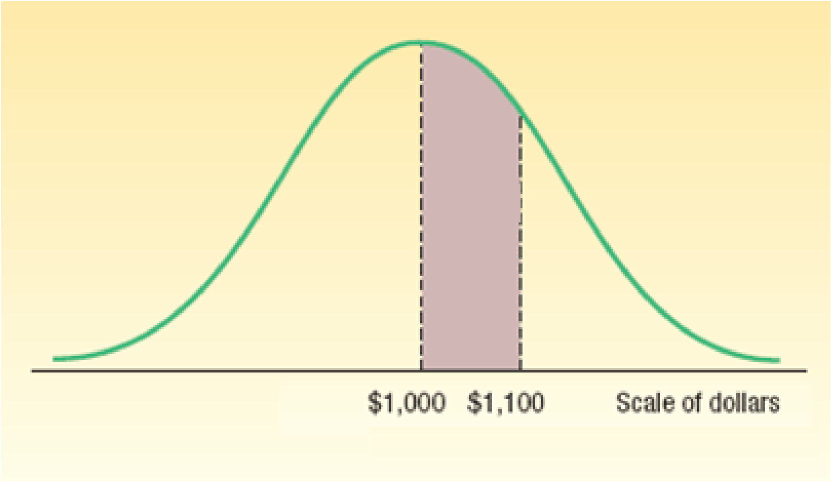
Normal Distribution – Finding Probabilities

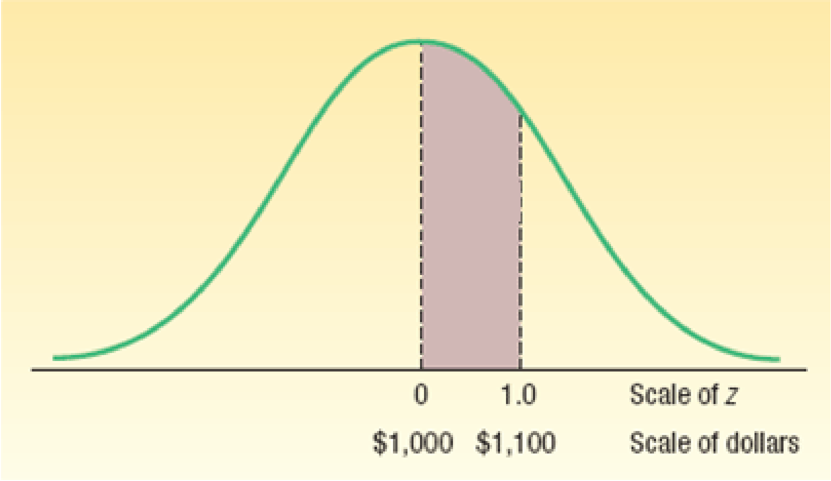
In the previous example we reported that the mean weekly income of a shift foreman in the glass industry is normally distributed with a mean of $1,000 and a standard deviation of $100.

What is the likelihood of selecting a foreman whose weekly income is between $1,000 and $1,100?



Normal Distribution – Finding Probabilities





For X = \$1,000:

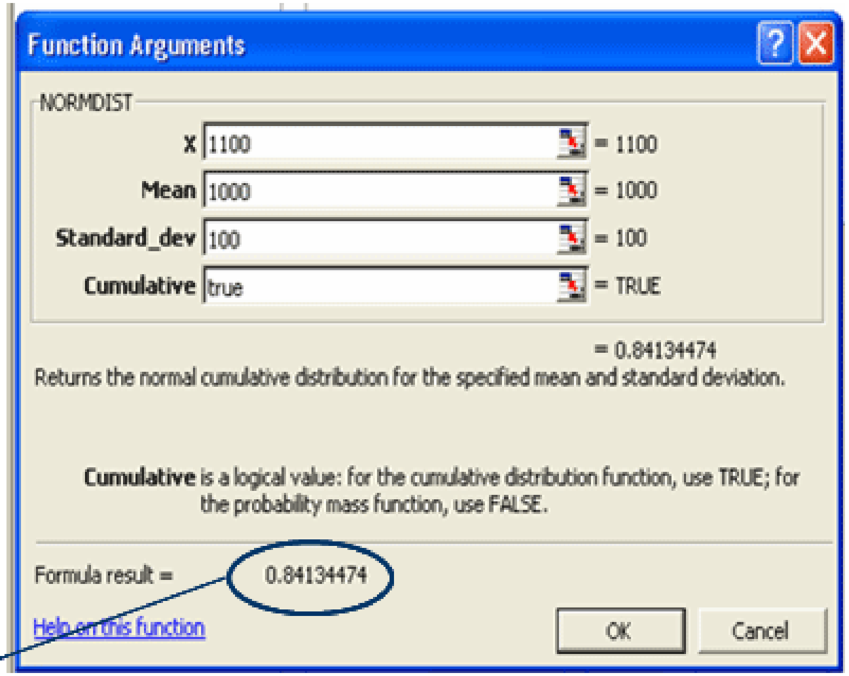
$$z = \frac{X-\mu}{\sigma} = \frac{\$1{,}000 - \$1{,}000}{\$100} = 0.00$$

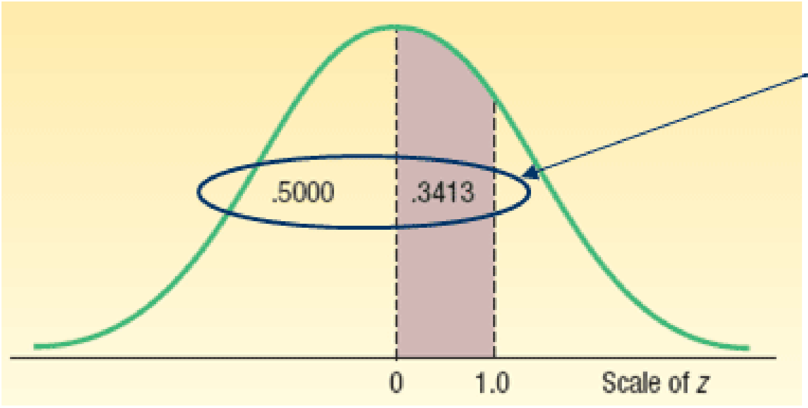
For X = \$1,100:

$$z = \frac{X-\mu}{\sigma} = \frac{\$1{,}100 - \$1{,}000}{\$100} = 1.00$$

Finding Areas for Z Using Excel

The Excel function
=NORMDIST(x,Mean,Standard_dev,Cumu)
=NORMDIST(1100,1000,100,true)
generates area (probability) from
Z=1 and below





Normal Distribution – Finding Probabilities (Example 2)

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of \$1,000 and a standard deviation of \$100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

Between \$790 and \$1,000?

LO5

Normal Distribution – Finding Probabilities (Example 2)

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of \$1,000 and a standard deviation of \$100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

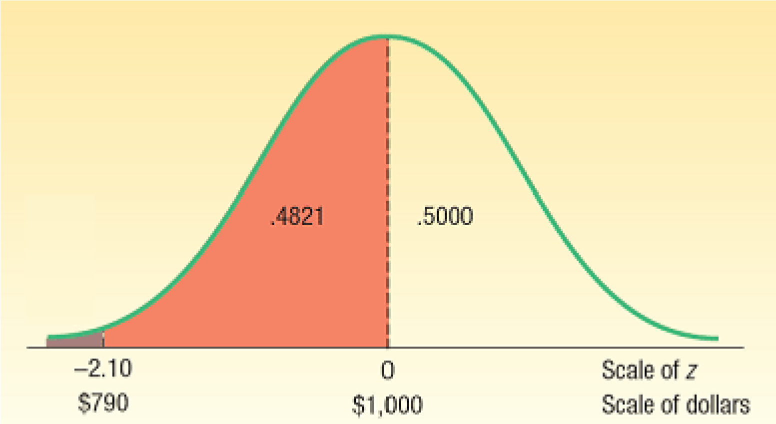
Between \$790 and \$1,000?

For $X = \$790$:

$$z = \frac{X-\mu}{\sigma} = \frac{\$790 - \$1{,}000}{\$100} = -2.10$$

For $X = \$1{,}000$:

$$z = \frac{X-\mu}{\sigma} = \frac{\$1{,}000 - \$1{,}000}{\$100} = 0.00$$



Normal Distribution – Finding Probabilities (Example 3)

LO5

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of $1,000 and a standard deviation of $100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

Less than $790?

Normal Distribution – Finding Probabilities (Example 3)

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of $1,000 and a standard deviation of $100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

Less than $790?

Find Z for X = $790 :

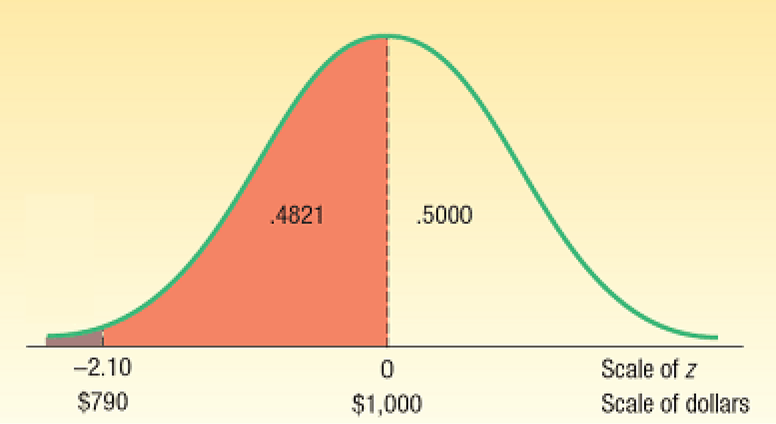
$$z = \frac{X-\mu}{\sigma} = \frac{\$790 - \$1{,}000}{\$100} = -2.10$$

To find the are below - 2.10,

subtract from 0.50 the area from - 2.10 to 0

= 0.50-0.4821

= 0.0179



Normal Distribution – Finding Probabilities (Example 4)

LO5

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of $1,000 and a standard deviation of $100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

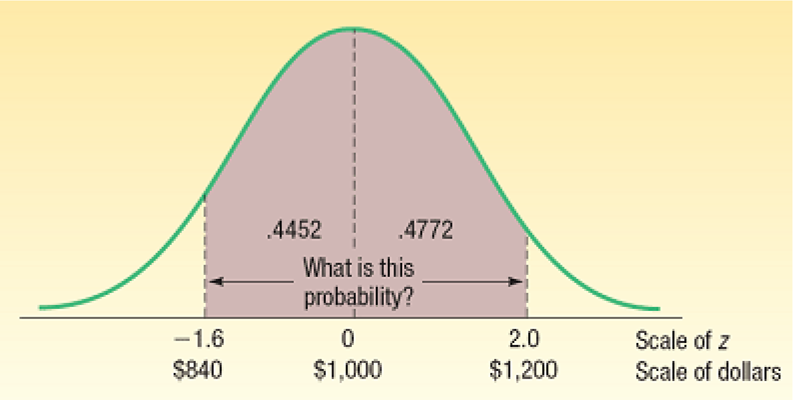
Between $840 and $1,200?

For $X = \$840$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$840 - \$1,000}{\$100} = -1.60$$

For $X = \$1,200$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1,200 - \$1,000}{\$100} = 2.00$$



LO5

Normal Distribution – Finding Probabilities (Example 5)

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of $1,000 and a standard deviation of $100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

Between $1,150 and $1,250

Normal Distribution – Finding Probabilities (Example 5)

LO5

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of \$1,000 and a standard deviation of \$100.

What is the probability of selecting a shift foreman in the glass industry whose income is:

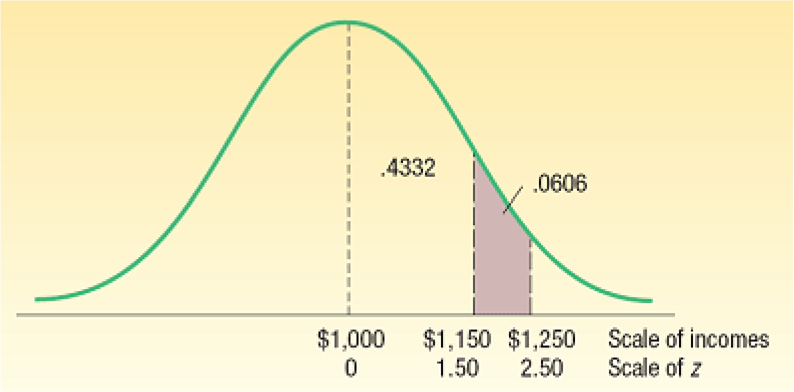
Between \$1,150 and \$1,250

For X = \$1,150:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1{,}150 - \$1{,}000}{\$100} = 1.50$$

For X = \$1,250:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1{,}250 - \$1{,}000}{\$100} = 2.50$$



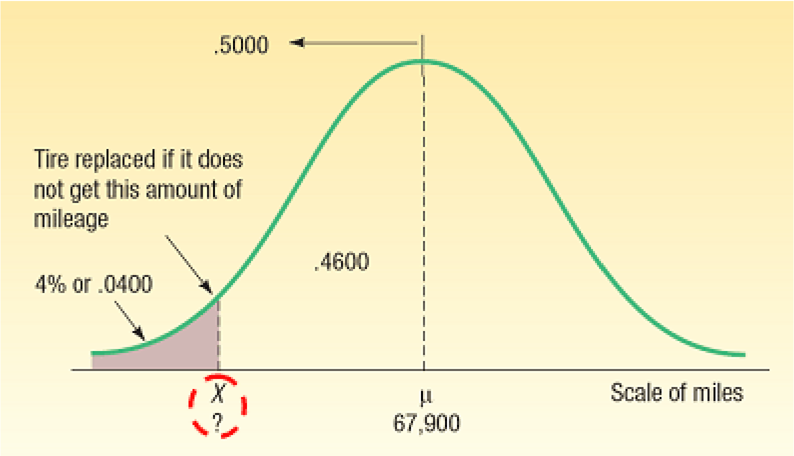
Using Z in Finding X Given Area - Example

Layton Tire and Rubber Company wishes to set a minimum mileage guarantee on its new MX100 tire. Tests reveal the mean mileage is 67,900 with a standard deviation of 2,050 miles and that the distribution of miles follows the normal probability distribution. Layton wants to set the minimum guaranteed mileage so that no more than 4 percent of the tires will have to be replaced.

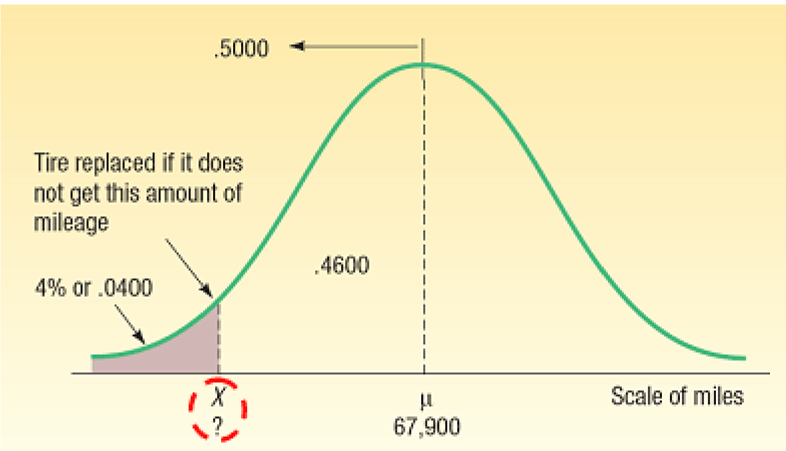
What minimum guaranteed mileage should Layton announce?

LO5

Using Z in Finding X Given Area - Example



Using Z in Finding X Given Area - Example



Solve X using the formula:

$$z = \frac{x - \mu}{\sigma} = \frac{x - 67{,}900}{2{,}050}$$

The value of z is found using the 4% information

The area between 67,900 and x is 0.4600, found by 0.5000 - 0.0400

Using Appendix B.1, the area closest to 0.4600 is 0.4599, which gives a z alue of -1.75. Then substituting into the equation:

$$-1.75 = \frac{x - 67{,}900}{2{,}050}, \text{ then solving for x}$$

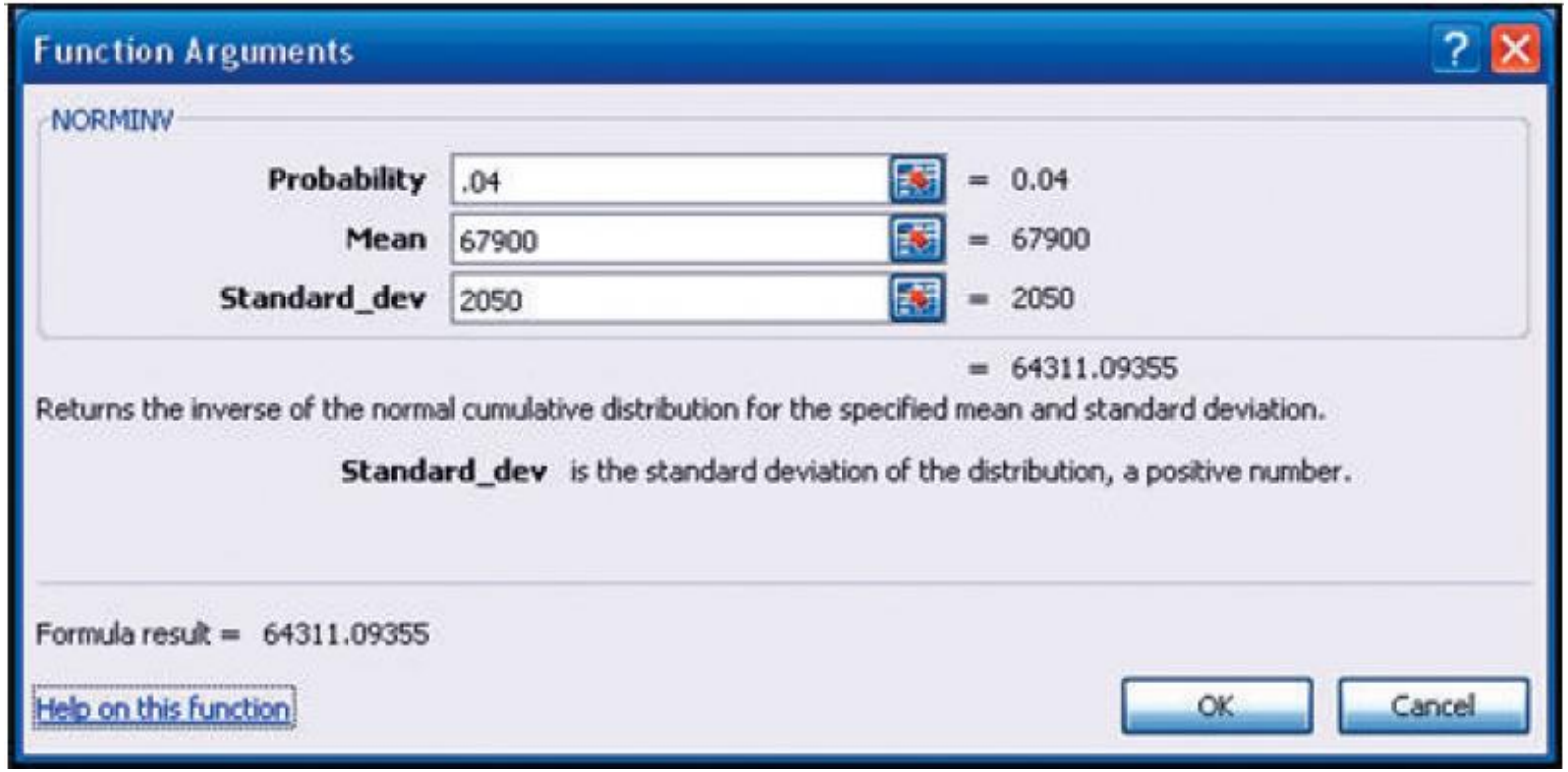
$$-1.75(2{,}050) = x - 67{,}900$$

$$x = 67{,}900 - 1.75(2{,}050)$$

$$x = 64{,}312$$

LO5

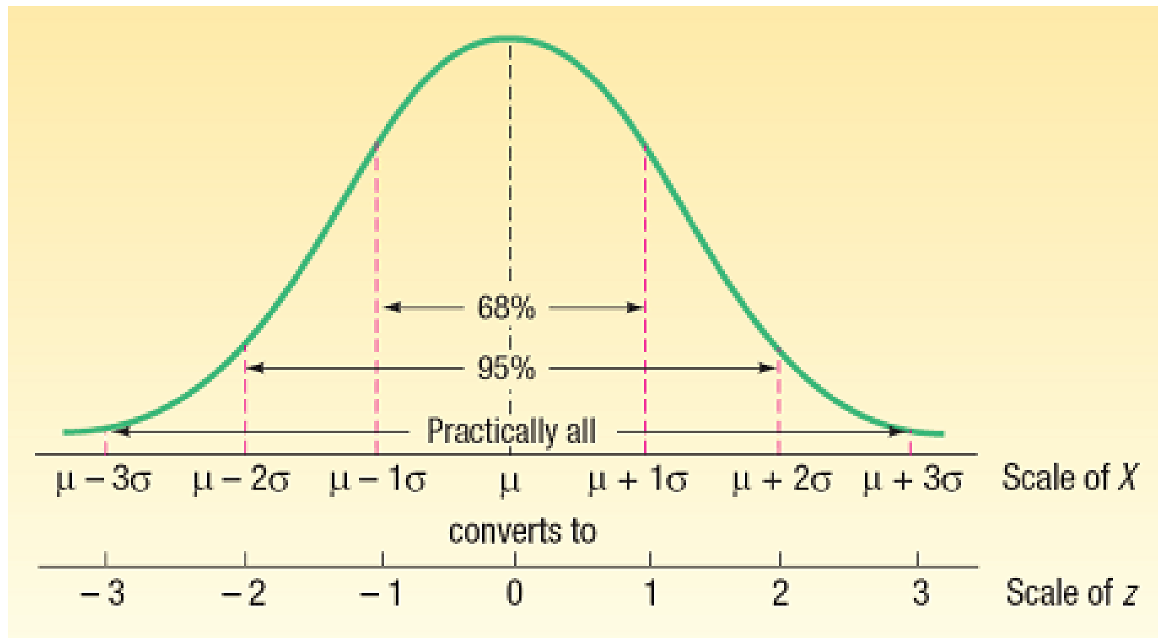
Using Z in Finding X Given Area - Excel



LO6 Find probabilities using the Empirical Rule.

The Empirical Rule

- About **68 percent** of the area under the normal curve is within **one standard deviation** of the mean.
- About **95 percent** is within **two standard deviations** of the mean.
- **Practically all** is within **three standard deviations** of the mean.



LO6

The Empirical Rule - Example

As part of its quality assurance program, the Autolite Battery Company conducts tests on battery life. For a particular D-cell alkaline battery, the mean life is 19 hours. The useful life of the battery follows a normal distribution with a standard deviation of 1.2 hours.

Answer the following questions.

1. About 68 percent of the batteries failed between what two values?
2. About 95 percent of the batteries failed between what two values?
3. Virtually all of the batteries failed between what two values?

We can use the results of the Empirical Rule to answer these questions.

1. About 68 percent of the batteries will fail between 17.8 and 20.2 hours by 19.0 ± 1(1.2) hours.
2. About 95 percent of the batteries will fail between 16.6 and 21.4 hours by 19.0 ± 2(1.2) hours.
3. Virtually all failed between 15.4 and 22.6 hours, found by 19.0 ± 3(1.2)

This information is summarized on the following chart.